

# Combinatorial Concepts With Sudoku

## I: Symmetry

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### Abstract

The immensely popular logic game Sudoku incorporates a significant number of combinatorial concepts. In this essay, I use the mathematical and recreational study of Sudoku as a motivating example to introduce these concepts. Note that I *do not* consider techniques or strategies for *playing* Sudoku as these are extensively covered elsewhere.

## 1 Symmetry in Sudoku

Many of the Sudoku puzzles published in newspapers and elsewhere have the property that the initial pattern of non-empty squares is *symmetric*.

Most people would intuitively agree that the pattern of clues in the puzzle in Figure 1 is highly symmetric, but what exactly does *symmetric* mean and how can we quantify the different types of symmetry that occur in Sudoku puzzles? The precise answer to these questions leads us to a branch of mathematics called *group theory*.

	2			3			4	
6								3
		4				5		
			8		6			
8				1				6
			7		5			
		7				6		
4								8
	3			4			2	

Figure 1: Symmetric Sudoku

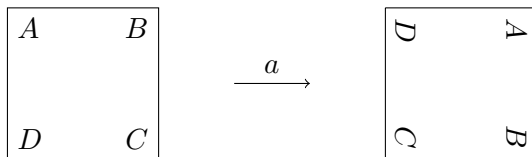
	4			8			6	
3								2
		7				4		
			7		8			
4				1				3
			5		6			
		6				5		
2								4
	8			6			3	

Figure 2: Rotated Symmetric Sudoku

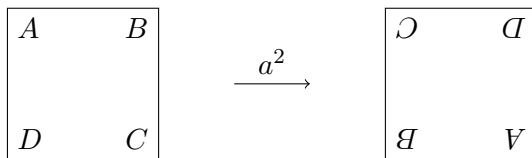
A shape or pattern is called *symmetric* if it can be transformed in some way without changing the way that it looks; for example we can *rotate* the puzzle of Figure 1 by  $90^\circ$  clockwise and, as shown in Figure 2, the pattern of clues looks the same.

So what are the possible transformations that can be symmetries of a Sudoku puzzle? Clearly the transformation must leave the overall square shape of the grid unchanged, so we can start by considering what operations do this — in technical language we are going to identify the *symmetries of a square*.

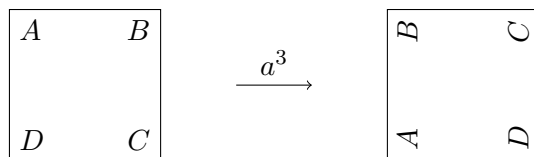
We have already seen that one of these symmetries is *rotation by  $90^\circ$* —we’ll give this operation a name and call it  $a$ .



Now clearly we can *repeat* this operation, rotating by another  $90^\circ$ , thereby getting a  $180^\circ$  rotation in total. As this operation arises by performing  $a$  twice, we’ll call it  $aa$  or just  $a^2$ .

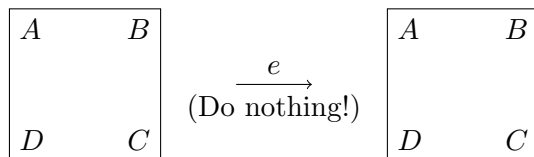


We can repeat this once more, getting a  $270^\circ$  rotation that, unsurprisingly, we will call  $a^3$ .



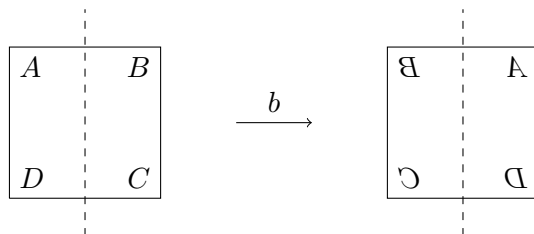
What happens if we do *another*  $90^\circ$  rotation? This corresponds to doing a full  $360^\circ$  rotation, which is the same as doing nothing at all. Should we have a name for the operation “doing nothing”? Although it may seem superfluous, it turns out to be useful to give this operation a name, and so we’ll give it the special name  $e$ , and we have the equation

$$a^4 = e.$$

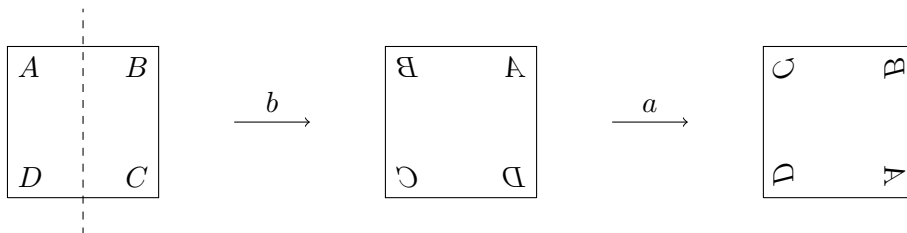


We say that  $a$  has *order* 4 because applying it 4 times brings the square back to its original position *for the first time*. Obviously applying it 8 times, or 12 times, or 20 times will also bring the square back to its original position, but not for the first time. The  $180^\circ$  rotation  $a^2$  has order 2 because we only need to apply that twice in order to return to the original position. You can check that  $a^3$  has the same order as  $a$ .

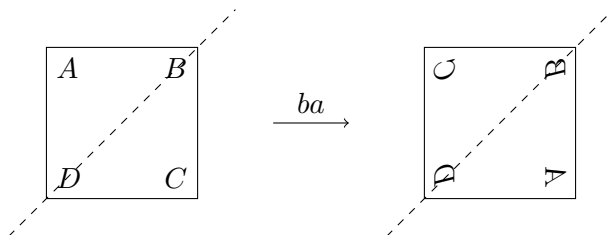
As well as the rotations, certain *reflections* are also symmetries of the square. For example, we can reflect the square through a vertical line, as shown below. We will call this reflection operation  $b$ .



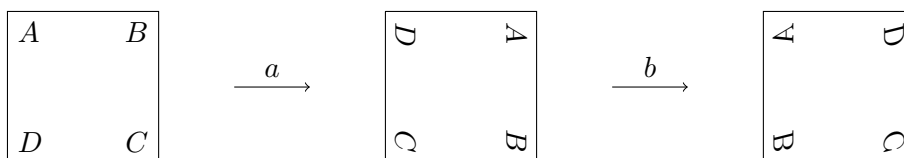
One property of symmetries that we used above is that if you *combine* two or more symmetries, then you get another one. For example, what happens if we *first* do a reflection  $b$  and then the rotation  $a$ ?



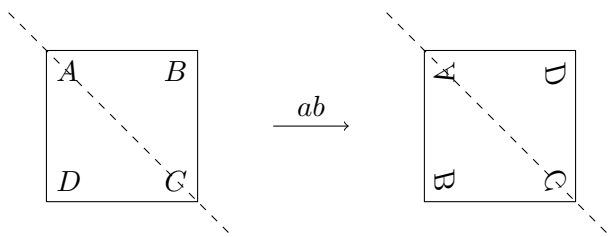
This combined operation, which is called  $ba$ , has the same effect as a *diagonal reflection* through the diagonal line running from the bottom-left to the top-right of the square; this is known as the *anti-diagonal*.



Notice that if we do  $a$  first and *then*  $b$ , we do not get the same result!



In fact,  $ab$  is equal to the *other* diagonal reflection, this time in the *main diagonal*.



It is easy to verify that this diagonal reflection can also be obtained by first doing  $b$  and then  $a^3$ , so we get the equation

$$ab = ba^3.$$

Are there any more symmetries of the square? There is obviously at least one more missing, which is the reflection through a horizontal axis; it will come as no surprise to discover that this can be obtained as the combination  $ba^2$ .

So now, we have 8 symmetries of the square as follows:

Name	Symmetry	Order
$e$	Identity (do nothing)	1
$a$	90° clockwise rotation	4
$a^2$	180° rotation	2
$a^3$	270° clockwise rotation	4
$b$	Reflection in vertical axis	2
$ba$	Reflection in anti-diagonal	2
$ba^2$	Reflection in horizontal axis	2
$ba^3$	Reflection in main diagonal	2

Now we can work out fairly quickly that no matter what combination of these symmetries we apply to the square, there is no way of getting a new one that is not on the list. For example, we might consider what happens if we first reflect in the horizontal axis ( $ba^2$ ) and then in the main diagonal ( $ba^3$ ). The combined operation is then

$$ba^2ba^3 = ba^2(ab) = ba^3b = (ab)b = ae = a$$

where we twice used the fact that  $ba^3 = ab$  and then that  $b^2 = e$ .

In fact, it turns out that these are *all* of the symmetries of the square and that any combination of these symmetries is already in the list — this means that the collection of 8 symmetries listed above forms a *group*.

#### DEFINITION

A *group* is a set  $G$  together with a binary operation  $\cdot$  and a special element  $e \in G$  satisfying the following properties:

- [CLOSURE] If  $x, y \in G$  then  $x \cdot y \in G$
- [ASSOCIATIVITY] For all  $x, y, z \in G$  we have  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- [IDENTITY] For all  $x \in G$  we have  $x \cdot e = e \cdot x = x$
- [INVERSES] For each  $x \in G$  there is an element  $x^{-1} \in G$  such that  $x \cdot x^{-1} = x^{-1} \cdot x = e$

Normally we simply use  $xy$  to represent  $x \cdot y$ .

For the symmetries of a square, we have already been using the idea of combining two symmetries by performing one after the other, and this is the binary operation for this group. This group contains 8 elements and is called the *dihedral group* of order 8, usually denoted  $D_8$  (though some authors denote it  $D_4$  just to confuse everyone).

For this group of symmetries, the *inverse* of an element  $x$  is essentially the transformation that “undoes” whatever  $x$  did. For example the inverse of the 90° rotation  $a$  is the 270° rotation  $a^3$ , because doing these two rotations one after another (in either order) is the same as doing nothing. In words we say that “the inverse of  $a$  is the element  $a^3$ ” or in symbols that

$$a^{-1} = a^3.$$

Now a Sudoku puzzle may happen to have all of these operations as symmetries (like the one in Figure 1) but it is also possible for a puzzle to have just *some* of them; for example Figure 3 shows a puzzle whose *only* non-identity symmetry is reflection in the main diagonal<sup>1</sup>. We assume that *every* puzzle has the identity or “do-nothing” operation as a symmetry, and so the full list of symmetries for this puzzle is  $\{e, ba^3\}$ .

							1	2
		3		4	5			
			6		1		7	
		4				6		
		5	8					
				3		4		
	1		2					
	7							

Figure 3: A puzzle with  $ba^3$  as its only non-identity symmetry

If puzzles such as the ones in Figure 4 and Figure 5 has the rotation  $a$  as a symmetry, then they *must* also have  $a^2$  and  $a^3$  (and of course  $e$ ) as symmetries, and so we cannot find a puzzle with  $a$  as its *only* non-identity symmetry. Similarly if a puzzle has  $x$  and  $y$  as symmetries, then it must also have  $xy$  as a symmetry. This means that the set of symmetries of any puzzle must *themselves* form a group, which is called a *subgroup* of  $D_8$ .

DEFINITION

A subset  $H$  of a group  $G$  is called a *subgroup* of  $G$  if the elements of  $H$  together with the binary operation  $\cdot$  and identity element  $e$  from  $G$  form a group.

	8		5					
	9	6	7				2	4
							9	
							7	2
3	1							
	4							
2	5				3	8	1	
					8		3	

Figure 4: A puzzle with symmetry group  $\{e, a, a^2, a^3\}$

<sup>1</sup>Although I found this puzzle, it was first put into this form by “Red Ed” from the [www.sudoku.com](http://www.sudoku.com) forums.

					2		
	8			7		9	
6		2			5		
	7			6			
			9	1			
				2		4	
		5			6		3
	9		4			7	
		6					

Figure 5: Another puzzle with symmetry group  $\{e, a, a^2, a^3\}$

The *order* of a group is the number of elements in the group, and it is a standard result in group theory that the order of a subgroup must be a divisor of the order of the group. Therefore seeing  $D_8$  has order 8, it follows that any Sudoku puzzle must have exactly 1, 2, 4 or 8 symmetries. A subgroup of order 1 must consist only of the single element  $\{e\}$  while a subgroup of order 2 must consist of  $e$  together with one of the 5 possible elements of order 2 (namely  $a^2$ ,  $b$ ,  $ba$ ,  $ba^2$  or  $ba^3$ ) and the subgroup of order 8 must be the entire dihedral group  $D_8$ .

We have already seen that it is possible to have a symmetry group of order 4 consisting only of rotations (Figure 4 and Figure 5). And if a subgroup contains either  $a$  or  $a^3$ , then it must contain this entire subgroup. So, are there any other subgroups of order 4? Such a subgroup must contain  $e$  and three other elements that cannot include  $a$  or  $a^3$ ; a quick calculation shows that it cannot contain three elements from  $\{b, ba, ba^2, ba^3\}$  because if it did, then some combination of them would be equal to  $a$ . Therefore a subgroup of order 4 must contain  $e$ ,  $a^2$  and exactly two elements from the set  $\{b, ba, ba^2, ba^3\}$ . It is then easy to see that there are precisely two other possibilities for a group of order 4, which are

$$\{e, a^2, b, ba^2\} \quad \text{and} \quad \{e, a^2, ba, ba^3\}.$$

We can find Sudoku puzzles with exactly these symmetry groups, and these are shown in Figure 6 and Figure 7 (warning: this one is *hard* to solve).

Now, it is easy to see that if we have a puzzle with a reflection symmetry, for example  $b$ , then by transposing the matrix (that is, swapping rows and columns) we immediately get a puzzle with the reflection symmetry  $ba^2$ . So in some sense a horizontal reflection is the same “type” of symmetry as a vertical reflection. We can make this concept precise by using the group theoretic notion of *conjugacy*; two elements  $x$  and  $y$  in a group  $G$  are *conjugate* if

$$y = z^{-1}xz$$

for some  $z \in G$ . For example,  $ba^2$  is conjugate to  $b$  because (using  $a$  as the conjugating element  $z$ ) we get

$$a^{-1}(ba^2)a = a^3ba^3 = baa^3 = b.$$

5							4
9							1
	8		3		4		9
	6					4	
			2		6		
	9					3	
	7		9		5	6	
2							5
1							9

Figure 6: A puzzle with symmetry group  $\{e, b, a^2, ba^2\}$

6							3
	7			8			9
		2				5	
			3				
	8			1			7
					2		
		5				1	
	9			4			8
3							2

Figure 7: A puzzle with symmetry group  $\{e, ba, a^2, ba^3\}$



Therefore, treating conjugate subgroups of  $D_8$  as representing the same type of symmetry, we end up with the following 7 distinct possibilities for a symmetric Sudoku puzzle:

Type	Order	Description	Group
Type I	8	Full dihedral symmetry	$D_8$
Type II	4	Full rotational symmetry	$\{e, a, a^2, a^3\}$
Type III	4	Horizontal and vertical reflection	$\{e, b, a^2, ba^2\}$
Type IV	4	Diagonal and anti-diagonal reflection	$\{e, ba, a^2, ba^3\}$
Type V	2	$180^\circ$ rotational symmetry	$\{e, a^2\}$
Type VI	2	Horizontal <i>or</i> vertical reflection	$\{e, b\}$ or $\{e, ba^2\}$
Type VII	2	Diagonal <i>or</i> anti-diagonal reflection	$\{e, ba\}$ or $\{e, ba^3\}$

## VERSION HISTORY

Version 0.2: 2006-03-29: Updated based on comments from Adam Glessner and typos spotted by Glenn Fowler.

Version 0.1: 2006-03-28: Originally Posted to [www.sudoku.com](http://www.sudoku.com).