Permanents and Determinants of Latin Squares

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Abstract: **Let L be a latin square of indeterminates. We explore the determinant and permanent of L and show that a number of properties of L can be recovered from the polynomials $\det(L)$ and $\operatorname{per}(L)$. For example, it is possible to tell how many transversals L has from $\operatorname{per}(L)$, and the number of 2×2 latin subsquares in L can be determined from both $\det(L)$ and $\operatorname{per}(L)$. More generally, we can diagnose from $\det(L)$ or $\operatorname{per}(L)$ the lengths of all symbol cycles. These cycle lengths provide a formula for the coefficient of each monomial in $\det(L)$ and $\operatorname{per}(L)$ that involves only two different indeterminates. Latin squares A and B are *trisotopic* if B can be obtained from A by permuting rows, permuting columns, permuting symbols, and/or transposing. We show that nontrisotopic latin squares with equal permanents and equal determinants exist for all orders $n \geq 9$ that are divisible by 3. We also show that the permanent, together with knowledge of the identity element, distinguishes Cayley tables of finite groups from each other. A similar result for determinants was already known.** © 2014 Wiley Periodicals, Inc. J. Combin. Designs 24: 132–148, 2016

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1. INTRODUCTION

Let $[n] = \{0, 1, 2, \ldots, n-1\}$ and let $X_n = \{x_i : i \in [n]\}$ be a set of n commuting indeterminates. A latin square, $L = [a_{ij}]$, of order n is an $n \times n$ matrix of symbols in

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which each symbol occurs exactly once in each row and exactly once in each column. In this paper, we exclusively consider latin squares with symbols from X_n, that is, $n \times n$ arrays in which each row and column is a permutation of X_n. Taking the determinant or permanent of such a latin square yields a homogeneous multivariate polynomial of degree n. Our aim is to investigate which latin square properties can be determined solely from these polynomials. Examples of such properties might be the number of transversals (selections of n different symbols from different rows and columns), or the number of intercalates (subsquares of order 2). Determination of such properties may provide a broad stratification of latin squares to be used in searches or classifications. Additional motivation comes from [1] and [2], where permanents and determinants of latin squares of indeterminates have been used to prove nontrivial general properties of latin squares.

Let $\mathcal{S}_n$ be the set of permutations of $[n]$. The composition of permutations $\psi, \phi \in \mathcal{S}_n$, written $\psi\phi$, is taken to be the permutation $x \mapsto \psi\phi(x) = \psi(\phi(x))$. For a latin square $L = [a_{ij}]$, of order n, the *permanent* of L is defined by

$$per(L) = \sum_{\mu \in \mathcal{S}_n} a_{0\mu(0)} a_{1\mu(1)} \cdots a_{n-1\mu(n-1)}$$

and the *determinant* of L is defined by

$$\det(L) = \sum_{\mu \in \mathcal{S}_n} \epsilon(\mu) a_{0\mu(0)} a_{1\mu(1)} \cdots a_{n-1\mu(n-1)},$$

where

$$\epsilon(\mu) = \begin{cases} 1, & \text{if } \mu \text{ is an even permutation,} \\ -1, & \text{if } \mu \text{ is an odd permutation.} \end{cases}$$

When considering polynomials we will always assume that like terms have been collected. Both per(L) and det(L) can be thought of as a sum of *monomials* where each monomial is the product of indeterminates, with an integer coefficient. A *bivariate monomial* is a monomial of the form $zx_e^u x_f^v$, where $u + v = n$ and z is an integer. To simplify the language in proofs, we allow u, v or z to equal 0. In §5 we identify the coefficient of every bivariate monomial in per(L) and det(L).

Two polynomials F and G in the indeterminates X_n are said to be *similar* if there exists a permutation $\sigma \in \mathcal{S}_n$ such that $F(x_0, \dots, x_{n-1}) = \pm G(x_{\sigma(0)}, \dots, x_{\sigma(n-1)})$. In other words, F and G are similar if one can be transformed into the other by relabeling variables and/or by multiplying by -1. The polynomials F and G are *dissimilar* if they are not similar. Similarity of determinants (or likewise permanents) induces an equivalence relation on latin squares. One aim of this paper is to better understand the equivalence classes under this relation.

Two latin squares L and M are *isotopic* if one can be obtained from the other by permuting rows, permuting columns, and/or permuting symbols. We say that L and M are *trisotopic* if L is isotopic to M or isotopic to the transpose M^T of M. The set of latin squares that are trisotopic to L is the *trisotopy class* of L. It is immediate from the definitions that two latin squares in the same trisotopy class must have similar determinants and similar permanents. More interestingly, in §6 we will see examples of general families of latin squares from different trisotopy classes that have similar determinants and similar permanents. This will allow us to answer several questions posed by Ford and Johnson [9], who investigated latin squares with similar determinants.

For latin squares that are isotopic to the Cayley table of a finite group, Formanek and Sibley [10] and Mansfield [15], showed that the determinant determines the group. We show a similar result for permanents in §3.

In §4, we provide details of trisotopy class invariants which we used to verify that two latin squares of order at most eight have similar permanents if and only if they belong to the same trisotopy class. However, in §6 we show that the same statement fails for order 9. The ideas behind this paper go back to the foundation of group representation theory. For full details see [6, 11, 14], but the brief summary is this: Frobenius, after prompting by Dedekind, invented group characters en route to describing the factorization of the group determinant. A group matrix is similar to a block diagonal matrix in which each block is indecomposable and corresponds to an irreducible representation. The character corresponding to a block is essentially the first nonzero coefficient of the characteristic polynomial of the block. Modern work has examined further information contained in the group determinant. However, even in the group case, only a small proportion of this information has been exploited thus far. This paper investigates how generalizing the ideas of Frobenius et al. leads to invariants that tie in with combinatorial and algebraic properties of quasigroups. We hope that this lays a theoretical foundation for further results along the lines of those proved in [1] and [2].

2. NOTATION AND TERMINOLOGY

In this section, we collect some notation and terminology that will be used throughout the paper.

It is assumed throughout that L denotes a latin square of order n. The rows and columns of L will be indexed by $[n]$. By a *diagonal* of L, we will mean any set of n cells from different rows and different columns of L. The permanent and determinant are both defined in terms of the products of entries on diagonals.

For each latin square $L = [a_{ij}]$, of order n, and each $k \in [n]$ we define $\theta_k \in \mathcal{S}_n$ by $\theta_k(i) = j$ if $a_{ij} = x_k$. Now, for each ordered pair $(e, f) \in [n] \times [n]$ define $\theta_{e,f} \in \mathcal{S}_n$ by $\theta_{e,f} = \theta_f \theta_e^{-1}$. For $i \in [n]$, we find that $\theta_{e,f}(i)$ is the index of the column that contains x_f in the same row in which x_e occurs in column i. We say that $\theta_{e,f}$ is the *symbol permutation* corresponding to the pair (e, f). As $\theta_{e,f}$ is a derangement it can be written as a product of disjoint cycles,

$$\theta_{e,f} = \theta_{e,f}^{(1)} \theta_{e,f}^{(2)} \theta_{e,f}^{(3)} \cdots \theta_{e,f}^{(q)}, \tag{1}$$

each of length at least 2. Let the length of the cycle $\theta_{e,f}^{(i)}$ be denoted by ℓ_i, so that we have $\ell_1 + \ell_2 + \ldots + \ell_q = n$. Corresponding to each $\theta_{e,f}^{(i)}$ is a *symbol cycle* consisting of the $2\ell_i$ occurrences of the symbols x_e, x_f in the columns permuted by $\theta_{e,f}^{(i)}$. Following the convention adopted in [19] and used in many papers since, we say that the *length* of the symbol cycle corresponding to $\theta_{e,f}^{(i)}$ is ℓ_i. The lengths of the cycles $\theta_{e,f}^{(1)}, \theta_{e,f}^{(2)}, \ldots, \theta_{e,f}^{(q)}$ can be used to define a sequence $\ell_{e,f} = (\ell_1, \ell_2, \ldots, \ell_q)$ where it is assumed that $\ell_j \leq \ell_{j+1}$, for all $j \in [q-1]$.

Similarly, for each $r \in [n]$ we may define a permutation $\rho_r = \rho_r(L)$ of $[n]$ by $\rho_r(i) = j$ if and only if $a_{ri} = x_j$. We then define, for each $r, s \in [n]$, a *row permutation* $\rho_{r,s}(L)$ of $[n]$ by $\rho_{r,s} = \rho_s \rho_r^{-1}$. Again, for any two rows the corresponding row permutation can be thought of as a composition of disjoint cycles that correspond to *row cycles*. Similarly, for any two columns we may define a *column permutation* on the set $[n]$, which is a composition of disjoint cycles that correspond to *column cycles*. Note that if two latin squares, A and B, are isotopic then there is a correspondence between row permutations (column permutations) of A and the row permutations (column permutations) of B. For a detailed study of the row, column, and symbol cycles in small latin squares, see [20].

We can associate $L = [a_{ij}]$ with a quasigroup $([n], \circ)$ in which $i \circ j = k \Leftrightarrow a_{ij} = x_k$. With this interpretation, $\theta_k(L)$ is the permutation $k/j \mapsto j$ for all $j \in [n]$, where $/$ denotes right division in the quasigroup. Also, $\rho_r(L)$ is the left translation by element r (i.e. multiplication on the left by r). Similarly $\rho_r(L^T)$ is right translation by r. The group generated by $\{\rho_r(L) : r \in [n]\} \cup \{\rho_r(L^T) : r \in [n]\}$ is known as the *multiplication group* of L, or *full mapping group* of L, and denoted $\mathrm{Mlt}(L)$.

As mentioned in the introduction, *isotopism* involves permuting the rows, permuting the columns and permuting the symbols of a latin square. Hence, isotopism can be viewed as an action of $\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n$ on latin squares of order n. The stabilizer of a latin square under this action is its *autotopism group*. See [17] for a detailed study of autotopisms. There are two other well-known group actions on latin squares. The group $\mathcal{S}_3$ takes a latin square to 6 *conjugate* latin squares. Combining this action with isotopism gives an action of the wreath product $\mathcal{S}_n \wr \mathcal{S}_3$ on latin squares. The stabilizer of a latin square L under this action is called the *autoparatopism group* of L, which we denote $\mathrm{Par}(L)$.

3. CAYLEY TABLES OF GROUPS

Throughout this section, we interpret a Cayley table for any group to be a latin square of indeterminates. We prove that the permanent of a Cayley table determines the group, provided the identity element is known.

Let G be a group of order $n \geq 3$, and let ε be the identity element of G. The group matrix $\mathcal{M}_G$ is the latin square of the operation of right division; it has $x_{gh^{-1}}$ in cell (g, h) for each $g, h \in G$. By permuting columns, it is obvious that $\mathcal{M}_G$ is isotopic to the usual Cayley table of G, which has x_{gh} in cell (g, h). Consider the coefficient ϕ_G of x_ε^{n-3} in $\mathrm{per}(\mathcal{M}_G)$. Observe that ϕ_G is a homogeneous polynomial of degree 3. In the following discussion, it is not assumed that g, h, k are distinct.

Lemma 3.1. *Suppose g, h, k are nonidentity elements of G. The monomial $x_g x_h x_k$ appears in ϕ_G if and only if $ghk = \varepsilon$ and/or $gkh = \varepsilon$. The monomial $x_g x_h x_\varepsilon$ appears in ϕ_G if and only if $h = g^{-1}$.*

Proof. Suppose $x_g x_h x_k$ appears as a monomial in ϕ_G, with $g, h, k \neq \varepsilon$. After applying the same suitably chosen permutation to both the rows and columns, the matrix $\mathcal{M}_G$ may be partitioned as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is of dimensions $(n-3)\times(n-3)$ and

$$D = \begin{pmatrix} x_\varepsilon & x_\alpha & x_\lambda \\ x_\mu & x_\varepsilon & x_\beta \\ x_\gamma & x_\nu & x_\varepsilon \end{pmatrix},$$

where $\{\alpha,\beta,\gamma\}=\{g,h,k\}$ as multisets and $\lambda,\mu,\nu\in G$ (we may ignore the possibility that $\{\lambda,\mu,\nu\}=\{g,h,k\}$ since in that case swapping the last pair of rows and swapping the last pair of columns puts D into the claimed form).

It follows that each monomial in

$$x_\varepsilon^{n-3}per(D) = x_\varepsilon^{n-3}\big(x_\varepsilon^3 + x_\varepsilon(x_\alpha x_\mu + x_\beta x_\nu + x_\lambda x_\gamma) + x_\alpha x_\beta x_\gamma + x_\lambda x_\mu x_\nu\big), \tag{2}$$

appears in per($\mathcal{M}_G$) with a nonzero coefficient (not less than its coefficient in (2)). Suppose with the ordering indicated, that the last three rows and columns of $\mathcal{M}_G$ are indexed by p,q,r. It follows that $pq^{-1}=\alpha$, $qr^{-1}=\beta$, and $rp^{-1}=\gamma$, and that $pr^{-1}=\lambda$, $qp^{-1}=\mu$, $rq^{-1}=\nu$. This implies that either $ghk=\varepsilon$ or $gkh=\varepsilon$.

Now suppose that $ghk=\varepsilon$. Consider the partition of $\mathcal{M}_G$ given above where the last three rows and columns of $\mathcal{M}_G$ are indexed, for any $z\in G$, by $p=z$, $q=g^{-1}z$, and $r=h^{-1}g^{-1}z$. It follows that $pq^{-1}=g$, $qr^{-1}=h$, and $rp^{-1}=k$ and $x_gx_hx_k$ is a monomial in ϕ_G.

It remains to consider monomials of the form $x_gx_hx_\varepsilon$. Suppose that the last three rows of $\mathcal{M}_G$ are indexed by p,q,r. If $x_gx_hx_\varepsilon$ occurs in per(D) then the pair $\{g,h\}$ must be equal to one of the pairs $\{\alpha,\mu\}$, $\{\beta,\nu\}$, or $\{\lambda,\gamma\}$. Now $\alpha=pq^{-1}=(qp^{-1})^{-1}=\mu^{-1}$, and similarly $\beta=\nu^{-1}$ and $\lambda=\gamma^{-1}$, so it follows that $g^{-1}=h$.

Conversely, suppose that $g=h^{-1}$. If z is any element of G then if $q=z$ and $r=g^{-1}z$ it follows that $qr^{-1}=g$ and $rq^{-1}=h$, and hence $x_gx_hx_\varepsilon$ appears as a monomial in ϕ_G. The Lemma is proved. □

From Lemma 3.1, per($\mathcal{M}_G$) contains the information to determine all triples (g,h,k) such that $gh=k^{-1}$ or $hg=k^{-1}$. Also, per($\mathcal{M}_G$) contains the information to determine k^{-1} from k. It follows that for each pair of elements (g,h) the set $\{gh,hg\}$ is known.

The following Lemma has appeared in several places (see, for example, [16]). Recall that a bijection f between two groups is an anti-isomorphism if $f(gh)=f(h)f(g)$ for all g,h.

Lemma 3.2. *If G, H are finite groups and there is a bijection $f: G\to H$ such that for all $g,h\in G$ we have that $f(gh)$ is either $f(g)f(h)$ or $f(h)f(g)$, then f is either an isomorphism or an anti-isomorphism.*

We can now prove the main result for this section.

Theorem 3.3. *Let L_G and L_H be Cayley tables for two finite groups G and H with respective identity elements ε_G and ε_H. If per(L_G) is similar to per(L_H) via a bijection that maps ε_G to ε_H, then G is isomorphic to H.*

Proof. Since the Cayley table of a group can be obtained by permuting the columns of the group matrix for the group, we may assume that per($\mathcal{M}_G$) is similar to per($\mathcal{M}_H$) via a bijection f such that $f(\varepsilon_G)=\varepsilon_H$. As before, let ϕ_G (resp. ϕ_H) be the coefficient of $x_{\varepsilon_G}^{n-3}$ in per($\mathcal{M}_G$) (resp. $x_{\varepsilon_H}^{n-3}$ in per($\mathcal{M}_H$)). By Lemma 3.1, $h=g^{-1}$ in G if and only if $x_g x_h x_{\varepsilon_G}$ appears in ϕ_G, which happens if and only if $x_{f(g)} x_{f(h)} x_{\varepsilon_H}$ occurs in ϕ_H. Thus $f(g^{-1}) = f(g)^{-1}$.

Also, a monomial $x_g x_h x_k$ occurs in ϕ_G if and only if the monomial $a x_{f(g)} x_{f(h)} x_{f(k)}$ occurs in ϕ_H. Now suppose that $ghk = \varepsilon_G$, i.e. $gh = k^{-1}$. Then by Lemma 3.1 the monomial $x_g x_h x_k$ occurs in ϕ_G and thus $x_{f(g)} x_{f(h)} x_{f(k)}$ occurs in ϕ_H. Therefore, again using Lemma 1, $f(g)f(h)f(k) = \varepsilon_H$ and/or $f(g)f(k)f(h) = \varepsilon_H$. In the first case $f(g)f(h) = f(k)^{-1} = f(k^{-1}) = f(gh)$. In the second case $f(h)f(g) = f(k)^{-1} = f(k^{-1}) = f(gh)$. Thus f satisfies the conditions of Lemma 2 and is either an isomorphism or an anti-isomorphism. Since anti-isomorphic groups are isomorphic, the theorem follows. □

It is quite plausible that Theorem 3.3 is still true without the hypothesis about the identity elements. However, note that the identity element of a group cannot be deduced solely from the permanent of a Cayley table for the group. To see this, define an operation $\oplus$ on $[n]$ by $x \oplus y \equiv x + y - 1 \bmod n$. Compared to the usual addition mod n, this new operation gives a group with the same permanent but a different identity element.

4. SMALL ORDER DATA

In this section, we discuss how well the determinant and permanent distinguish trisotopy classes of latin squares of small order. First we consider how many different monomials are contained in these polynomials. We assume, of course, that like terms have been collected and that monomials are counted only if they have a nonzero coefficient.

Theorem 4.1. *If L is a latin square of order $n > 2$, then per(L) and* $\det(L)$ *each contain no more than*

$$\binom{2n-1}{n} - n(n-1) \tag{3}$$

different monomials.

Proof. Chang [5] showed that a diagonal of L can contain any combination of symbols, with the exception that it is impossible to have $n-1$ occurrences of one symbol and 1 occurrence of some different symbol. Suppose that on a given diagonal there are u_i occurrences of symbol i for each $i \in [n]$. The number of choices for n nonnegative integers $u_1, \ldots, u_n$ with $u_1 + u_2 + \cdots + u_n = n$ is $\binom{2n-1}{n}$ (this is a standard combinatorial problem, equivalent to the number of ways to place n unlabeled balls in n labeled boxes). However, by Chang's Theorem we cannot have $u_i = n-1$ and $u_j = 1$ for $i \neq j$. Assuming $n > 2$, there are $n(n-1)$ ways to choose the pair (i, j) in this scenario. The result follows. □

As a consequence, we see that per(L) must have some monomials with coefficients that are super-exponentially large. The coefficients sum to $n!$, so the largest must be at least

$$\frac{n!}{\binom{2n-1}{n}} = \frac{n!^2(n-1)!}{(2n-1)!} = 2\sqrt{2}\pi\left(\frac{n}{4e}\right)^n (n + O(1))$$

as $n \to \infty$, by Stirling's formula.

TABLE I. Range of the number of monomials in determinants and permanents of latin squares of small order. Only partial enumerations were performed for the asterisked orders.

Order	Fewest in det	Fewest in per	Mean in det	Mean in per	Most in det	Most in per	Bound from (3)
2	2	2	2	2	2	2	n/a
3	4	4	4	4	4	4	4
4	10	10	11	11	11	11	23
5	26	26	54	54	82	82	106
6	68	80	226	241	367	397	432
7	246	246	1,224	1,351	1,310	1,436	1,674
8	810	810	5,174	5,759	5,491	6,054	6,379
9*	2,704	2,704			21,662	23,604	24,238
10*	7,492	9,252			86,188	91,273	92,288
11*	32,066	32,066			338,916	351,038	352,606

In Table I, we give data on the number of monomials in det(L) and per(L) as L ranges across all latin squares of order n, for each $n \leq 8$. We show the fewest number, average number and maximum number of monomials seen in each polynomial. For $9 \leq n \leq 11$ we show similar data, except that our enumeration was only partial. To be precise, we checked the latin squares with large autotopism groups. This includes all Cayley tables of groups, which tended to have far fewer monomials than any other latin squares. In particular, the cyclic group had the fewest monomials in every case that we surveyed. It remains possible that there exists a latin square with fewer monomials than the fewest listed in Table 1, or with more monomials than the most given in the table for $9 \leq n \leq 11$. We did not calculate averages for these partial enumerations. Table 1 also lists the bound obtained from (3). It is achieved for $n = 3$ but not for larger n, although the data hints that it might be asymptotically tight, at least for the permanent.

The following latin squares achieve the maximum number of monomials in per(L) for latin squares L of order 7 and 8, respectively.

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_2 & x_1 & x_4 & x_3 & x_6 & x_7 & x_5 \\ x_3 & x_4 & x_5 & x_6 & x_7 & x_1 & x_2 \\ x_4 & x_5 & x_1 & x_7 & x_3 & x_2 & x_6 \\ x_5 & x_7 & x_6 & x_1 & x_2 & x_4 & x_3 \\ x_6 & x_3 & x_7 & x_2 & x_1 & x_5 & x_4 \\ x_7 & x_6 & x_2 & x_5 & x_4 & x_3 & x_1 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_3 & x_6 & x_2 & x_4 & x_7 & x_8 & x_5 \\ x_7 & x_1 & x_4 & x_8 & x_2 & x_5 & x_6 & x_3 \\ x_5 & x_8 & x_1 & x_3 & x_6 & x_2 & x_7 & x_4 \\ x_8 & x_7 & x_2 & x_1 & x_5 & x_4 & x_3 & x_6 \\ x_2 & x_6 & x_8 & x_5 & x_1 & x_3 & x_4 & x_7 \\ x_6 & x_2 & x_7 & x_4 & x_3 & x_1 & x_5 & x_8 \\ x_3 & x_4 & x_5 & x_7 & x_8 & x_6 & x_1 & x_2 \\ x_4 & x_5 & x_3 & x_6 & x_7 & x_8 & x_2 & x_1 \end{pmatrix}.$$

They have autotopism groups of order 1 and 24, respectively. Unusually for latin squares of order 8, the one given above has no transversals. Hence, its permanent and determinant contain no monomial of the form $zx_1x_2x_3x_4x_5x_6x_7x_8$. Its determinant contains 5,214 monomials, some way short of the maximum achieved by latin squares of order 8. Clearly, there is some cancellation occurring in the determinant for this square.

In contrast, we see in Table 1 that the fewest monomials in det(L) often matches the fewest monomials in per(L), indicating a lack of cancellation in these cases.

Next, we move on to the main topic of this section, which is the question of how well det and per distinguish trisotopy classes of small order. We start with the determinant. For latin squares of order 1,2,3, all latin squares belong to the same trisotopy class. Hence in each case all squares of the same order have similar determinants. For orders 4 and 5, respectively, there are two trisotopy classes and in each case nontrisotopic latin squares have dissimilar determinants. For order 6, there are 17 trisotopy classes and for order 7, there are 324 trisotopy classes and again, nontrisotopic latin squares have dissimilar determinants.

For order 8, Ford and Johnson [9] found 842,227 trisotopy classes and all but 37 of these are a similarity class on their own, with respect to determinants. Further, the 37 *exceptional* trisotopy classes partitioned into 12 equivalence classes, say $\mathcal{C}_i$, for $i = 1, \ldots, 12$, where latin squares in the same equivalence class have similar determinants. Moreover in each of these 37 exceptional trisotopy classes the latin squares can be written as the union of four disjoint subsquares of order 4. For example, they showed that the following two latin squares have similar determinants, even though they are not trisotopic:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 & x_0 & x_3 & x_2 & x_5 & x_4 & x_7 & x_6 \\ x_2 & x_3 & x_0 & x_1 & x_6 & x_7 & x_4 & x_5 \\ x_3 & x_2 & x_1 & x_0 & x_7 & x_6 & x_5 & x_4 \\ x_4 & x_5 & x_7 & x_6 & x_0 & x_3 & x_2 & x_1 \\ x_5 & x_4 & x_6 & x_7 & x_1 & x_2 & x_3 & x_0 \\ x_7 & x_6 & x_4 & x_5 & x_3 & x_0 & x_1 & x_2 \\ x_6 & x_7 & x_5 & x_4 & x_2 & x_1 & x_0 & x_3 \end{pmatrix} \quad \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 & x_0 & x_3 & x_2 & x_5 & x_4 & x_7 & x_6 \\ x_2 & x_3 & x_0 & x_1 & x_6 & x_7 & x_4 & x_5 \\ x_3 & x_2 & x_1 & x_0 & x_7 & x_6 & x_5 & x_4 \\ x_4 & x_5 & x_7 & x_6 & x_0 & x_3 & x_2 & x_1 \\ x_5 & x_4 & x_6 & x_7 & x_2 & x_1 & x_0 & x_3 \\ x_7 & x_6 & x_4 & x_5 & x_1 & x_2 & x_3 & x_0 \\ x_6 & x_7 & x_5 & x_4 & x_3 & x_0 & x_1 & x_2 \end{pmatrix}. \tag{4}$$

We next investigate the situation for permanents of small latin squares. Consider a set of trisotopy class representatives of orders up to eight. It is not a simple matter to calculate the permanent of each of these latin squares and compare them pairwise for similarity (each polynomial typically has thousands of monomials, as discussed at the start of the section). Instead we computed some simpler trisotopy class invariants that would be equal for any two squares with equal permanents. The two invariants we computed for all latin squares of order up to eight are as follows.

Invariant #1: For each of the $n!$ diagonals of a latin square L, count the occurrences of each symbol on the diagonal. Sort these counts to produce a partition π of n. For every partition π of n, count the number d_π of diagonals of L that realize π. The vector (d_π) indexed by the partitions π is our first invariant.

Invariant #2: For each symbol s in the latin square L, find the lengths of all symbol cycles involving s. Suppose there are v_i cycles of length i, for $2 \leq i \leq n$. Since $v_{n-1} = 0$, and v_n is determined by the values of v_i for $i \leq n-2$ we define $\bar{v}_s = (v_2, v_3, \ldots, v_{n-2})$. We then sort the list of vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ lexicographically, and this becomes our second invariant.

We will also sometimes refer to the following invariant.

Invariant #3: Consider a multivariate polynomial such as det(L) or per(L). For each i, let v_i denote the number of monomials that contain positive powers of exactly i distinct indeterminates. The vector $(v_1, \ldots, v_n)$ is our third invariant. Note that before calculating v_i we collect all like terms.

Note that the number of transversals corresponds to the count d_π where π is the partition $1 + 1 + \cdots + 1$. The number of transversals is not a determinant invariant (and hence neither is Invariant #1). For example, consider the two latin squares in (4). They have similar determinants, but the right hand square has 320 transversals whereas the square on the left has none.

We discovered that any two trisotopy classes of order at most seven can be distinguished by at least one of Invariants #1 or #2, and thus they all have dissimilar permanents. For order 8 these two invariants do not completely distinguish trisotopy classes. However, there are only two sets of three trisotopy classes and 31 pairs that cannot be distinguished by either invariant. Happily, these numbers are sufficiently small that it was practical to compute the permanents of each of the trisotopy class representatives and compare them. Invariant #3 could separate some but not all of the examples. However, an approach that invariably allowed us to distinguish a target pair (A, B) of trisotopy class representatives was to choose a set I of eight distinct integers. We then evaluated per(A) by substituting I for the values of the variables $x_0, \ldots, x_7$ and evaluated per(B) at all possible permutations of I. If the value for per(B) never matched the value first computed for per(A) then we knew per(A) and per(B) were dissimilar. It turns out that this happened in every case for the first I that we chose. We conclude that:

Theorem 4.2. *Two latin squares of order at most eight have the same permanent if and only if they belong to the same trisotopy class.*

The same is not true for order 9 as will be shown later, by Corollary 6.3.

5. BIVARIATE MONOMIALS

For any pair $(e, f) \in [n] \times [n]$, the permanent per$(L)$ and determinant det(L) both contain bivariate monomials of the form $zx_e^u x_f^v$ where $u + v = n$ and z is a nonzero integer. Such bivariate monomials always exist, for instance when $u = 0$ and $v = n$. The main aim of this section is to characterize the bivariate monomials that occur, and to determine the coefficient z in each case. The characterization will be in terms of the lengths of symbol cycles. We first use these cycle lengths to define a *profile* for each pair of elements $(e, f) \in [n] \times [n]$. We write $S_L(e, f) = (\ell_1^{t_1}, \ldots, \ell_p^{t_p})$ where $\ell_j < \ell_{j+1}$ for $1 \le j < p$, and t_i records the number of times ℓ_i occurs in the sequence $\ell_{e,f}$. Clearly, $t_1\ell_1 + \ldots + t_p\ell_p = n$. We call $S_L(e, f)$ the *profile of pair* (e, f) with respect to the latin square L. It is worth recording that $S_L(e, f) = S_L(f, e)$.

Theorem 5.1. *Let $e, f \in [n]$ and let $S_L(e, f) = (\ell_1^{t_1}, \ldots, \ell_p^{t_p})$ be the profile of pair (e, f) with respect to the latin square L. If $0 \le u \le n$, then per(L) contains the monomial $zx_e^u x_f^{n-u}$, where*

$$z = \sum_{c_1,\ldots,c_p} \binom{t_1}{c_1}\binom{t_2}{c_2}\cdots\binom{t_p}{c_p} \tag{5}$$

and the sum is over nonnegative integers c_i satisfying $u = c_1\ell_1 + c_2\ell_2 + \ldots + c_p\ell_p$.

Proof. Consider the symbol cycles corresponding to (1), the decomposition of $\theta_{e,f}$. If we choose any subset, say U, of these symbol cycles we can find a diagonal of L by selecting the cells corresponding to every occurrence of x_e in the cycles in U and every occurrence of x_f in the cycles not in U. This diagonal will contribute to the coefficient of $x_e^u x_f^{n-u}$, where u is the sum of the lengths of the cycles in U. The sum in (5) counts the number of such choices that contribute to the coefficient of $x_e^u x_f^{n-u}$ given that, for each i, we choose some number c_i of the t_i cycles of length ℓ_i to form the set U.

It remains to show that these are the only contributions to the coefficient of $x_e^u x_f^{n-u}$. Consider any diagonal D that contributes to this coefficient.

Let Γ be a symbol cycle on the symbols x_e and x_f. Suppose D includes an occurrence of x_e in Γ and start at this entry. Now move to the x_f in the same row (which cannot be in D), and from there move to the x_e in the same column (which must be in D). Repeating this process, we find that D must include every x_e and no x_f from Γ.

Hence, within each symbol cycle on the symbols x_e and x_f, we see that D can include only one of the symbols. It follows that D was one of the contributions we had already counted, and we are done. □

As an example of Theorem 5.1, consider when $u = n$. In that case, the only nonzero contribution to (5) comes from taking $c_i = t_i$ for $1 \le i \le p$, which shows that $x_e^n x_f^0$ (that is, x_e^n) occurs with a coefficient of 1 in per(L). It is important to stress that (5) may evaluate to 0. For example, if $u = 1$ or $u = n - 1$ there is no way to partition u into cycle lengths from $\theta_{e,f}$, since the smallest possible cycle length is 2. Hence the coefficients of $x_e^1 x_f^{n-1}$ and $x_e^{n-1} x_f^1$ are both zero. This confirms what we saw in Theorem 4.1.

Theorem 5.2. *Let $e, f \in [n]$ and let $S_L(e, f) = (\ell_1^{t_1}, \ldots, \ell_p^{t_p})$ be the profile of pair (e, f) with respect to the latin square L. If $0 \le u \le n$, then* det(L) *contains the monomial $z x_e^u x_f^{n-u}$, where*

$$z = \epsilon(\theta_f) \sum_{c_1,\ldots,c_p} (-1)^w \binom{t_1}{c_1}\binom{t_2}{c_2}\cdots\binom{t_p}{c_p}, \tag{6}$$

the sum is over nonnegative integers c_i satisfying $u = c_1\ell_1 + c_2\ell_2 + \ldots + c_p\ell_p$, and $w = \sum c_i$ where the sum is over all i for which ℓ_i is even.

Proof. The proof is the same as for the previous theorem, but we need to keep track of the parity of the permutation that corresponds to each diagonal that contributes to the coefficient. Start with the permutation θ_f, which has parity $\epsilon(\theta_f)$. We then change it by choosing x_e rather than x_f from c_i symbol cycles of length ℓ_i, for $1 \le i \le p$. Each cycle of even length that we select changes the parity, while cycles of odd length make no difference. The result follows. □

The coefficient (6) is bounded in absolute value by (5). Hence, (6) will evaluate to zero whenever (5) does. In the case of the determinant it is also possible for a coefficient to vanish through cancellation. As a concrete example, consider $n = 8$, $u = 4$, and $S_L(e, f) = (2^2, 4^1)$. There are only two nonzero terms in the sum (6), corresponding to (c_1, c_2) being (2, 0) and (0, 1), respectively. The two terms are $+1$ and -1, so the result is that the coefficient of $x_e^4 x_f^4$ in this case would be zero.

Define the *profile* of a latin square L to be the multiset

$$S_L = \{S_L(e, f) : 1 \le e < f \le n\}.$$

We next show that the profile of L is determined by per(L), and also by det(L).

Theorem 5.3. *Given two latin squares L and M of order n, if per(L) is similar to per(M) then $S_L = S_M$. Likewise, if* $\det(L)$ *is similar to* $\det(M)$ *then $S_L = S_M$.*

Proof. Suppose that per(L) is similar to per(M). Since S_M is isotopy invariant we may, without loss of generality, permute the symbols of M so that per$(L) =$ per(M).

Suppose that $S_L \neq S_M$. Then there exist $e, f \in [n]$ for which $S_L(e, f) \neq S_M(e, f)$. Let j be the smallest value for which L and M have a different number of j-cycles in the symbol permutation for symbols x_e and x_f. Applying Theorem 5.1 in the case $u = j$ we find that the coefficient of $x_e^u x_f^{n-u}$ differs between per(L) and per(M) because precisely one term in (5) is different (namely the term where one c_i is one and all others are zero). This contradiction proves the first statement.

The second statement is proved similarly. By permuting the symbols of M and possibly interchanging two rows, we may assume that $\det(L) = \det(M)$. Now argue as above, but using Theorem 5.2. □

Regarding the above proof, we stress the importance of taking $u = j$ to be the *smallest* value with different numbers of j cycles. The conclusion may not be valid for some larger u. To see this, consider $u = n$: Even if $S_L(e, f)$ contains an n-cycle but $S_M(e, f)$ does not, both per(L) and per(M) will have the same coefficient of $x_e^u x_f^{n-u}$, namely 1.

The converses of the statements in Theorem 5.3 do not hold. There are examples of pairs of latin squares that have the same profile but do not have similar determinants or similar permanents. From [19] we know that there are 37 trisotopy classes of latin square of order 9 in which every symbol cycle has length 9. It is easy to establish that no two of these 37 classes have representatives whose permanents coincide, since they are distinguished by Invariant #1 from §4. The determinants do not coincide either. Among the 37 classes, there is only one pair whose determinants have the same number of monomials with nonzero coefficients. That pair differs using Invariant #3.

6. NONTRISOTOPIC PAIRS INDISTINGUISHABLE BY PER OR DET

In this section, we describe a method for constructing infinitely many pairs of latin squares that are not trisotopic but nevertheless have equal permanents and equal determinants. We start by discussing a type of latin square that we will need later.

A latin square L is said to be *all-even* if $\rho_r(L)$ is an even permutation for all rows r. It was shown by Häggkvist and Janssen [13] that all-even latin squares constitute no more than an exponentially small proportion of all latin squares. Despite this, they exist for all orders $n \geq 3$. Possibly the first person to show this was Ihringer [12]. The proof we offer here is our own.

Lemma 6.1. *For all $n \geq 3$ there exists an all-even latin square of order n.*

Proof. For odd n we simply use the Cayley table of the cyclic group: Every ρ_r is a power of a cycle of length n, and hence is even. For n divisible by 4, we use a latin square for which ρ_r consists of $n/2$ transpositions for all rows r except for one row where ρ_r is the identity. Such a latin square is guaranteed to exist by [21, Thm. 14], and it is obvious for this square that every ρ_r is an even permutation. For $n \equiv 2 \bmod 4$ we start by taking any latin square of the block form

$$\begin{pmatrix} E & F \\ F & E \end{pmatrix},$$

where E is an arbitrary latin square on the first $n/2$ symbols, and F is formed by replacing every entry x_i in E by $x_{i+n/2}$, for all $i \in [n/2]$. We then swap two columns within the copy of E in the bottom right corner (for this step, we require that $n > 2$). In the first $n/2$ rows, ρ_r consists of some cycles from the E block and cycles of matching lengths from the F block. Thus all cycle lengths occur an even number of times in ρ_r, so ρ_r is even. In the last $n/2$ rows, ρ_r differs from $\rho_{r-n/2}$ by $n/2 + 1$ transpositions, and hence is also even. □

Suppose L is an all-even latin square of order n. As the product of two even permutations, the row permutation $\rho_{r,s}(L)$ must be even, for all $r, s \in [n]$. Crucially for our purposes, this is an isotopy invariant property. In other words, any L' isotopic to L will have $\rho_{r,s}(L')$ being an even permutation for all $r, s \in [n]$. This result is not difficult to see directly by noting that $\rho_{r,s}$ is unchanged when we interchange two columns or two symbols. (It can also be inferred from a conjugate result to Theorem 5.3, since isotopic latin squares have similar permanents.)

Next, we introduce the structure that will play a key role in our main result for the section. Suppose $n = 3m$ for some integer $m > 1$. We say that a pair (A, B) of latin squares of order n is *suitable* if they satisfy the following conditions:

(1) A and B are both comprised of an $m \times m$ array of 3×3 blocks.
(2) The block in the top left corner of A is

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{pmatrix}. \tag{7}$$

(3) The block in the top left corner of B is the transpose of (7).
(4) A agrees with B in every other block, and all other blocks have the form

$$\begin{pmatrix} x_a & x_b & x_c \\ x_b & x_c & x_a \\ x_c & x_a & x_b \end{pmatrix} \tag{8}$$

for some $a, b, c \in [n]$.
(5) $\rho_i(A)$ is an even permutation for $i = 0, 3, 6, \ldots, n-3$.
(6) There exists $j \geq 3$ for which $\rho_0(A^T)$ and $\rho_j(A^T)$ have the same parity.

The entire structure of a suitable pair (A, B) is determined by specifying the rows of A indexed by $0, 3, 6, \ldots, n-3$. We will exploit this observation later when specifying suitable pairs. Our reason for being interested in such suitable pairs is as follows:

Theorem 6.2. *Let A and B be a suitable pair of latin squares of order $n > 3$. Then A and B are not trisotopic but nevertheless* $\operatorname{per}(A) = \operatorname{per}(B)$ *and* $\det(A) = \det(B)$.

Proof. Suppose $i \in \{0, 3, 6, \ldots, n-3\}$. Condition 5 tells us that $\rho_i(A)$ is even. Meanwhile Conditions 2 and 4 show that $\rho_{i,i+1}$ and $\rho_{i,i+2}$ consist of disjoint 3-cycles (which are even permutations), so $\rho_{i+1}(A)$ and $\rho_{i+2}(A)$ are both even. It follows that A is all-even. However, Conditions 3 and 4 show that $\rho_{0,3}(B)$ differs by one transposition from $\rho_{0,3}(A)$, and hence is odd. It follows that A and B are not isotopic. Similarly, Condition 6 ensures that $\rho_{0,j}(B^T)$ is odd and hence A is not isotopic to B^T either. In other words, A and B are not trisotopic.

Let V be the set of cells in the 3×3 block in the top left corner of A, and let V_0, V_1, V_2 be the set of cells in V that contain the symbols x_0, x_1, x_2 in A, respectively.

Define a permutation $\zeta = (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)\cdots(n-3\ n-2\ n-1)$ and let ξ be the permutation of $[n] \times [n]$ defined by $\xi(a, b) = (\zeta(a), \zeta^{-1}(b))$. By Condition 4, in both A and B, the cell $\xi(a, b)$ contains the same symbol as the cell (a, b), unless $(a, b) \in V$.

Consider the following involution Ω on the set of diagonals of A and B. For any diagonal D with one of the properties:

(a) D includes strictly more cells from V_1 than from V_2, and has no cells from V_0, or
(b) D includes a cell from V_0, a cell from V_2 and no cell from V_1,

we define $\Omega(D) = \xi(D)$ and $\Omega(\xi(D)) = D$. All other diagonals are defined to be fixed points of Ω.

We claim for all D that the product of the entries on D in A equals the product of the entries on $\Omega(D)$ in B. To test the validity of our claim, first note that it is trivial for all diagonals that have an equal number of entries from V_1 and V_2. They are precisely the diagonals that are fixed points of Ω. Any diagonal D moved by Ω is of one of four types:

- D has property (a): Then D contains cells in V_1 but not in V_2 or V_0, in which case Ω moves the cells in V_1 to V_2.
- $D = \Omega(D')$ where D' has property (a): Then D contains cells in V_2 but not in V_1 or V_0, in which case Ω moves the cells in V_2 to V_1.
- D has property (b): Then D contains a cell in V_0, a cell in V_2 and no cell in V_1, in which case Ω moves the cell in V_2 to V_0 and the cell in V_0 to V_1.
- $D = \Omega(D')$ where D' has property (b): Then D contains a cell in V_0, a cell in V_1 and no cell in V_2, in which case Ω moves the cell in V_1 to V_0 and the cell in V_0 to V_2.

The contents of cells outside V are unchanged by applying Ω and then changing from A to B. The cells inside V interchange x_1 and x_2 when moving from A to B. This proves our claim, which shows that $\mathrm{per}(A) = \mathrm{per}(B)$. The fact that $\det(A) = \det(B)$ follows by noting that the permutations corresponding to D and $\Omega(D)$ have the same sign. □

Corollary 6.3. *For all $m \geq 3$ there exists a pair (A, B) of nontrisotopic latin squares of order $n = 3m$ that satisfy* $\mathrm{per}(A) = \mathrm{per}(B)$ *and* $\det(A) = \det(B)$.

Proof. There are typically many ways to construct suitable pairs, but it suffices to give one.

By Lemma 6.1 there is a latin square M of order m for which $\rho_r(M)$ is even for all $r \in [m]$. Let L be the direct product of M with a cyclic group of order 3, which we form by replacing each entry, say x_i, of M by a subsquare of the form (8) with $a = 3i$, $b = 3i + 1, c = 3i + 2$. For each $r \in [m]$, our construction ensures that $\rho_{3r}(L)$ will have precisely nine inversions for every inversion in $\rho_r(M)$. Hence $\rho_{3r}(L)$ will be even, given that $\rho_r(M)$ is even. Now replace the top left block of L by (7) to get A, and by the transpose of (7) to get B. The first five conditions in the definition of suitable pairs will now be satisfied. If the sixth condition fails we do one more step. Let Y and Z denote the 3×3 blocks in rows indexed $\{0, 1, 2\}$ and columns indexed $\{3, 4, 5\}$ and $\{6, 7, 8\}$, respectively. Choose two symbols in Y and interchange them within Y, in both A and B. Do the same thing in Z. The overall effect is to change the parity of $\rho_r(A^T)$ for $3 \leq r \leq 8$ but to leave $\rho_r(A)$ unchanged for all $r \in [n]$. In this way, Condition 6 can be satisfied without spoiling any of the other conditions.

Applying Theorem 7 now gives the result. □

As an example, suppose m is odd. We may take $\rho_0(A) = (4\ 5)(7\ 8)$, which is an even permutation. For $i = 3, 6, 9, \ldots, n-3$ define row i of A by $a_{ij} = x_{i+j}$, where addition in the subscript is modulo n. Completing A and B in the obvious manner produces a suitable pair as constructed in Corollary 6.3.

We note that suitable pairs of latin squares of order $n = 3m$ intersect in $n^2 - 6$ cells and have the same permanent. It is not possible for a pair of latin squares of order n to intersect in $n^2 - 5$, $n^2 - 3$, $n^2 - 2$, or $n^2 - 1$ cells, see for example [3] or [8]. In the next result, we see that for $n > 2$ it is not possible to construct a pair of latin squares that intersect in $n^2 - 4$ cells and that have the same permanent.

Theorem 6.4. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two latin squares, of order $n > 2$, which intersect in $n^2 - 4$ cells. Then* $\mathrm{per}(A) \neq \mathrm{per}(B)$.

Proof. Assume without loss of generality that $a_{00} = a_{11} = x_0$, $a_{01} = a_{10} = x_1$ and $b_{00} = b_{11} = x_1$, $b_{01} = b_{10} = x_0$, but otherwise A and B agree in all remaining entries.

Then $\theta_{0,2}(A)$ differs from $\theta_{0,2}(B)$ by a transposition, and hence has different cycle structure. Let j be the smallest value for which $\theta_{0,2}(A)$ and $\theta_{0,2}(B)$ have a different number of cycles of length j. By (5), the coefficient of $x_0^j x_2^{n-2}$ is different in $\mathrm{per}(A)$ and $\mathrm{per}(B)$. Thus $\mathrm{per}(A) \neq \mathrm{per}(B)$. □

However, we might ask under what conditions is $\mathrm{per}(A)$ similar to $\mathrm{per}(B)$. Certainly it is similar whenever A and B are in the same trisotopy class. In [4] it was shown that for all $n \geq 5$ there exists a pair of isomorphic latin squares that differ in precisely four entries. Such pairs of squares are necessarily trisotopic. However, it is unclear if there are examples of two nontrisotopic latin squares, of the same order, which differ in four cells but have similar permanents. The above analysis also hold for determinants.

The following questions were asked at the end of [9].

(Q1) For which orders are there nontrisotopic squares with similar determinants?

(Q2) Are there nontrisotopic squares with trivial autoparatopism group with similar determinants?

(Q3) Are there nontrisotopic squares with full mapping group $\mathcal{S}_n$ with similar determinants?

Obviously, Corollary 6.3 gives a partial answer to Q1, by showing that such squares exist for all $n \geq 9$ satisfying $n \equiv 0 \bmod 3$. We can also answer Q2 and Q3, in the positive, with the following example of order 9. Consider the suitable pair (A, B) in which rows 0,3,6 of A are

$$[x_0\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8],\ [x_3\, x_4\, x_6\, x_0\, x_7\, x_8\, x_2\, x_5\, x_1],\ [x_5\, x_8\, x_7\, x_2\, x_1\, x_6\, x_0\, x_3\, x_4].$$

Both A and B have trivial autoparatopism group so, by Theorem 7, they answer Q2. They also answer Q3 since $\rho_{5,6}(A^T)$ and $\rho_{5,7}(A^T)$ generate $\mathcal{S}_9$, and columns 5,6,7 of A agree with the corresponding columns of B.

7. INVARIANTS

Over the course of our investigation thus far, we have encountered several properties that are invariant among latin squares with similar determinants or permanents. In this section, we summarize those observations and add some extra ones.

Suppose that A, B, C, and D are latin squares, that per(A) is similar to per(B), and det(C) is similar to det(D). Then

- A and B have the same order, since it is the degree of per(A) and per(B). Similarly, C and D have the same order, since it is the degree of det(C) and det(D).
- A and B have the same profile by Theorem 6. It follows that for each k they have the same number of symbol cycles of length k. The same is true for C and D.
- A and B have the same number of intercalates by the above, since intercalates are just symbol cycles of length 2. Similarly, C and D have the same number of intercalates.
- A and B need not have the same number of row/column cycles of length k, if $k > 2$. Similarly for C and D. Examples are easy to create using Theorem 7. For example, take $(A, B) = (C, D)$ to be the suitable pair of order 9 in which rows 0,3,6 of A are, respectively:

$$[x_0\, x_1\, x_2\, x_4\, x_3\, x_5\, x_7\, x_6\, x_8],\ [x_5\, x_8\, x_7\, x_2\, x_1\, x_6\, x_4\, x_0\, x_3],\ [x_6\, x_4\, x_3\, x_8\, x_7\, x_0\, x_2\, x_5\, x_1]. \quad (9)$$

  Here B has both row and column cycles of lengths 6 and 7, but A has no such cycles.
- A and B have the same number of transversals, since that number is the coefficient of $x_1x_2\cdots x_n$ in per(A) and per(B).
- C and D need not have the same number of transversals. Indeed, we cannot always tell from det(C) whether C has any transversals. For a concrete example, see (4).
- A and B need not have the same number of orthogonal mates. For example, consider the suitable pair (A, B) of order 9 in which rows 0,3,6 of A are

$$[x_0\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8],\ [x_3\, x_4\, x_6\, x_0\, x_7\, x_8\, x_2\, x_5\, x_1],\ [x_5\, x_8\, x_7\, x_1\, x_2\, x_6\, x_4\, x_3\, x_0].$$

  Here A has an orthogonal mate, and B has none.
- C and D need not have the same number of orthogonal mates. The previous example suffices. Also, (4) is a good example, since the right hand square has 12,048 orthogonal mates, while the left hand square has none.
- A and B have the same *symbol parity* since they have the same profile. It follows that they have the same *parity*, in the classical sense. Likewise C and D have the same symbol parity and the same parity. See [18, 20] for definitions of symbol parity and parity, and a full exploration of their importance.
- A and B need not have the same *row parity* or the same *column parity*. Indeed, if (A, B) are a suitable pair, then they necessarily have opposite row parity and opposite column parity. Similarly for C, D.
- A and B need not have autoparatopism groups of the same order. Similarly, C and D need not have autoparatopism groups of the same order. For example, consider the suitable pair (A, B) of order 9 in which rows 0,3,6 of A are

$$[x_0\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8],\ [x_3\, x_5\, x_4\, x_6\, x_7\, x_8\, x_0\, x_2\, x_1],\ [x_6\, x_8\, x_7\, x_0\, x_2\, x_1\, x_3\, x_4\, x_5].$$

In this case, A is not isotopic to any conjugate except itself. However, B is isotopic to its conjugate that exchanges rows with symbols. We have $|\mathrm{Par}(A)| = 12$ and $|\mathrm{Par}(B)| = 24$. Also consider the suitable pair (A, B) of order 9 in which rows 0,3,6 of A are

$$[x_0\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8],\ [x_3\, x_4\, x_5\, x_6\, x_7\, x_8\, x_0\, x_1\, x_2],\ [x_6\, x_8\, x_7\, x_0\, x_1\, x_2\, x_3\, x_5\, x_4].$$

Here A is isotopic to three of its six conjugates, but B is only isotopic to one of its conjugates (namely, itself). We have $|\mathrm{Par}(A)| = 2$ and $|\mathrm{Par}(B)| = 6$.

- A and B need not have full mapping groups of the same order. Similarly, C and D need not have full mapping groups of the same order. For example, take $(C, D) = (A, B)$ to be the suitable pair defined in (9). In this case, $\mathrm{Mlt}(A)$ is the alternating group A_9 and $\mathrm{Mlt}(B)$ is the symmetric group $\mathcal{S}_9$.

8. OPEN QUESTIONS

The examples we have encountered thus far did not settle the following questions.

- Are there nontrisotopic latin squares with equal determinants and equal permanents for some orders that are not multiples of 3?
- Are there latin squares with equal permanents but dissimilar determinants? We observed in §4 that there are latin squares of order 8 with equal determinants but dissimilar permanents.
- Are there two latin squares with equal permanents, but with autotopism groups of different orders?
- Are there two latin squares with equal permanents, but with different numbers of $k \times k$ subsquares for some $k > 2$? It follows from Theorem 6 that latin squares with equal permanents have the same number of 2×2 subsquares.
- Does there exist a latin square that is not isotopic to any group's Cayley table, but which nevertheless has the same permanent (or the same determinant) as the Cayley table of some group?
- Does Theorem 3.3 hold without the assumption that ε_G maps to ε_H?
- Is equality ever achieved in (3) for $n > 3$? For all large n are there examples achieving (3) to within a factor of $1 - o(1)$?
- Among the latin squares L of a given order n, does the Cayley table of the cyclic group always minimize the number of monomials in $\det(L)$ and $\mathrm{per}(L)$?

As explained in the introduction, the study of determinants of groups led to the birth of representation theory. It is interesting to consider which algebraic properties of quasigroups can be deduced from determinants of latin squares. For example, the fourth open question above relates to the issue of whether associativity can be deduced from the determinant. Also, Dickson [7] investigated the factorization of the group determinant modulo a prime p. He showed that the determinant of any p-group P of order n is equal modulo p to $(\sum_{q \in P} x_q)^n$. To what extent can this result be generalized to "group-like" quasigroups such as Moufang loops?

Meanwhile, the permanent of a latin square records that combinations of symbols can be found on the diagonals of the latin square. This natural combinatorial data contains a wealth of information about the latin square. This paper has taken the first step in showing how to extract some of that information.

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